2714 [June, 1918]. **Proposed by H. R. HOWARD, University of St. Francis Xavier's College, Nova Scotia.**

A shuffled pack of $2(p+q)$ cards contains $2p$ honors. Show that the chance of securing exactly half the honors in taking half the pack is $[F(p, q)]^2 \div F(2p, 2q)$, where $F(p, q)$ denotes the number of different sets of p cards which can be selected from $(p+q)$ cards.

Show also that if one honor is removed from the pack, the chance is not thereby affected. Is this true for the chance of getting any other assigned number of honors?

Solution by the Proposer.

We have to find first the number of ways in which we can get exactly p honors in $(p+q)$ cards.

We can obviously choose our "p" honors in $(2p)!/(p!\,p!)$ ways and the q other cards in $(2q)!/(q!\,q!)$ ways.

Hence, we can effect the required division in $[(2p)!\,(2q)!]/[(p!)^2\,(q!)^2]$ ways. Now the number of ways of taking $(p+q)$ cards from the full pack is $[2(p+q)]!/(p+q)!^2$. Thus the chance is

$$\frac{(2p)!\,(2q)!}{(p!)^2(q!)^2} \div \frac{[2(p+q)]!}{(p+q)!^2}, \tag{1}$$

i.e.,

$$\frac{(p+q)!^2}{(p!)^2\,(q!)^2} \div \frac{(2p+q)!}{(2p)!\,(2q)!},$$

i.e.,

$$[F(p, q)]^2 \div F(2p, 2q).$$

Now suppose one honor removed from the pack. Then the number of ways of taking $(p+q)$ cards from the remainder and obtaining exactly p honors is

$$\frac{(2p-1)!}{(p-1)!\,(p)!}\cdot\frac{(2q)!}{(q!)^2},$$

and the chance is

$$\frac{(2p-1)!}{(p-1)!\,(p)!}\,\frac{(2q)!}{(q!)^2} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{2}$$

Dividing (1) by (2) we obtain unity for the quotient and this proves their equality. Let x be the assigned number of honors. We shall show that the condition that the chances be equal can only be satisfied by $x = p$. With the full pack the chance will now be

$$\frac{(2p)!}{x!\,(2p-x)!}\cdot\frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q)!}{(p+q)!^2}. \tag{3}$$

With one honor removed the chance will be

$$\frac{(2p-1)!}{x!\,(2p-x-1)!}\,\frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{4}$$

Dividing (3) by (4) we see that these can only be equal if

$$\frac{p}{2p-x} = 1$$

or $x = p$, which proves our statement.

Also solved by H. L. Olson, A. Pelletier, and E. E. Witmer.

2715 [June, 1918]. **Proposed by H. R. KINGSTON, University of Manitoba.**

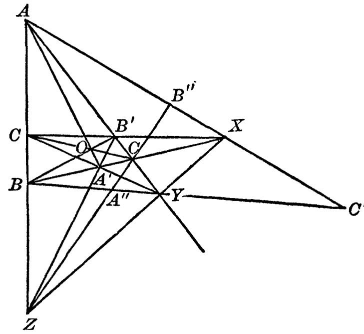
A', B', C' are points on the sides BC, CA, AB, respectively, of the triangle ABC, and AA', BB', CC' are concurrent in O. X, Y, Z are the three collinear points in which, by Desargues' theorem, the corresponding sides of the triangles ABC and $A'B'C'$ intersect. If A'', B'', C'' are the vertices of the triangle formed by the lines AX, BY, CZ, show that AA'', BB'', CC'' are concurrent.

I. Solution by William Hoover, Columbus, Ohio.

Take ABC as the triangle of reference, and the trilinear coördinates of the vertices A, B, C to be $(\alpha_2, 0, 0)$; $(0, \beta_2, 0)$; $(0, 0, \gamma_2)$, and those of O, $(\alpha_1, \beta_1, \gamma_1)$; then the equation of AO is $\gamma_1\beta - \beta_1\gamma = 0$.

The coördinates of A' will be proportional to $(0, \beta_1, \gamma_1)$, and, by symmetry, those of B' to $(\alpha_1, 0, \gamma_1)$, and of C' to $(\alpha_1, \beta_1, 0)$.

The equation of $A'B'$ is $\beta_1\gamma_1\alpha + \alpha_1\gamma_1\beta - \alpha_1\beta_1\gamma = 0$.



The point of intersection of AB and $A'B'$, or Z, has coördinates proportional to $(-\alpha_1, \beta_1, 0)$; those of the intersection BC and $B'C'$, or X, to $(0, -\beta_1, \gamma_1)$; and of CA and $C'A'$, or Y, to $(\alpha_1, 0, -\gamma_1)$; and X, Y, Z are collinear, since

$$\begin{vmatrix} 0, & -\beta_1, & \gamma_1 \\ \alpha_1, & 0, & -\gamma_1 \\ -\alpha_1, & \beta_1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & -1, & 1 \\ 1, & 0, & -1 \\ -1, & 1, & 0 \end{vmatrix}$$

$$= \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & 0, & 1 \\ 1, & -1, & -1 \\ -1, & 1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 1, & -1 \\ -1, & 1 \end{vmatrix} = 0.$$

The equation of AX is $\gamma_1\beta + \beta_1\gamma = 0$, and those of BY and CZ respectively $\gamma_1\alpha + \alpha_1\gamma = 0$, and $\beta_1\alpha + \alpha_1\beta = 0$.

The coördinates of C'', the point of intersection of AX and BY, are proportional to $(\alpha_1, \beta_1, \gamma_1)$; of BY and CZ, to $(\alpha_1, \beta_1, \gamma_1)$; and of CZ and AX to $(\alpha_1, \beta_1, \gamma_1)$, the latter two points being A'', B'' respectively.

It is evident now that the equations to AA'', BB'', CC'' are the same as those of AO, BO, CO, in order, the first three lines then passing through O.

II. Solution by H. L. Olson, Chicago, Illinois.

I shall amplify this theorem by proving that the lines AA'', BB'', CC'' are identical with the lines AA', BB', CC' respectively, and hence intersect in the point O. In the triangles $BC'Y$ and $CB'Z$, the lines BC, $C'B'$, and YZ, joining corresponding vertices, meet at X, and hence the points A, A', A'', in which corresponding sides meet, are collinear. Similarly B, B', B'' are collinear; also C, C', C''. Hence the lines AA'', BB'', CC'' meet at O.

Also solved by A. Pelletier and the Proposer.

2716 [June, 1918]. Proposed by Clifford N. Mills, Brookings, South Dakota.

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

Solution by E. H. Worthington, Elkins Park, Pa.

A velocity of 40 miles per hour is the same as $58\frac{2}{3}$ feet per second. If v is the velocity of the raindrop, we have $v = 58\frac{2}{3} \tan 20° = 58\frac{2}{3} \times 0.364$ feet per sec. $= 21.35$ feet per second.

Also solved by H. E. Carleton, A. M. Harding, H. L. Olson, A. Pelletier and J. B. Reynolds.

2735 [December, 1918]. Proposed by H. B. Phillips, Massachusetts Institute of Technology.

If two lines AE and BD, drawn from the vertices A and B of a triangle to the opposite sides, divide the angles A and B so that the parts of A are respectively less than the corresponding parts of B, then AE is greater than BD.